Near-best univariate spline discrete quasi-interpolants on non-uniform partitions

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Abstract. Univariate spline discrete quasi-interpolants (abbr. dQIs) are approximation operators using B-spline expansions with coefficients which are linear combinations of discrete values of the function to be approximated. When working with nonuniform partitions, the main challenge is to find dQIs which have both good approximation orders and bounded uniform norms independent of the given partition. Near-best dQIs are obtained by minimizing an upper bound of the infinite norm of dQIs depending on a certain number of free parameters, thus reducing this norm. This paper is devoted to the study of some families of near-best dQIs of approximation order 2.

§1.Introduction

A spline quasi-interpolant (abbr. QI) of f has the general form

$$Qf = \sum_{\alpha \in A} \mu_\alpha(f) B_\alpha$$

where $\{B_\alpha, \alpha \in A\}$ is a family of B-splines forming a partition of unity and $\{\mu_\alpha(f), \alpha \in A\}$ is a family of linear functionals which are local in the sense that they only use values of f in some neighbourhood of $\Sigma_\alpha = supp(B_\alpha)$. The main interest of QIs is that they provide good approximants of functions without solving any linear system of equations. In the literature, one can find the three following types of QIs:

(i) *Differential QIs* (abbr. DQIs) : the linear functionals are *linear combinations of values of derivatives* of f at some point in Σ_α (see e.g. [5-7]).
(ii) *Discrete QIs* (abbr. dQIs) : the linear functionals are *linear combinations of values* of f at some points in some neighbourhood of Σ_α (see e.g. [1-3], [6], [9], [11], [13], [15-16], [24]).
(iii) *Integral QIs* (abbr. iQIs) : the linear functionals are *linear combinations of weighted mean values* of f in some neighbourhood of Σ_α (see e.g. [2-3], [6], [13-14], [24-25]).

In this paper and a subsequent one, we shall study various types of univaraite dQIs and iQIs, more specifically those that we call *near-best* QIs which are defined as follows:

(dQIs) assume that $\mu_\alpha(f) = \sum_{\beta \in F_\alpha} \lambda_\alpha(\beta) f(x_\beta)$ where the finite set of points $\{x_\beta, \beta \in F_\alpha\}$ lies in some neighbourhood of Σ_α. Then it is clear that, for $\|f\|_\infty \le 1$ and $\alpha \in A$, $|\mu_\alpha(f)| \le \|\lambda_\alpha\|_1$, where λ_α is the vector with components $\lambda_\alpha(\beta)$, from which we deduce immediately

$$\|Q\|_\infty \le \sum_{\alpha \in A} |\mu_\alpha(f)| B_\alpha \le \max_{\alpha \in A} |\mu_\alpha(f)| \le \max_{\alpha \in A} \|\lambda_\alpha\|_1 = \nu_1(Q).$$

Now, assuming that $n = card(F_\alpha)$ for all α, we can try to find a $\lambda_\alpha^* \in \mathbb{R}^n$ solution of the minimization problem

$$\|\lambda_\alpha^*\|_1 = \min\{\|\lambda_\alpha\|_1;\ \lambda_\alpha \in \mathbb{R}^n,\ V_\alpha \lambda_\alpha = b_\alpha\}$$

where the linear constraints express that Q is exact on some subspace of polynomials. Thus, we finally obtain

$$\|Q\|_\infty \le \nu_1^*(Q) = \max_{\alpha \in A} \|\lambda_\alpha\|_1.$$

(iQIs) assume that $\mu_\alpha(f) = \sum_{\beta \in F_\alpha} \lambda_\alpha(\beta) \int_{\Sigma_\beta} M_\beta(t) f(t) dt$, where the B-splines M_β are normalized by $\int M_\beta = 1$. Once again, for $\|f\|_\infty \le 1$, we have

$$|\mu_\alpha(f)| \le \sum_{\beta \in F_\alpha} |\lambda_\alpha(\beta)| |\int_{\Sigma_\beta} M_\beta(t) f(t) dt| \le \sum_{\beta \in F_\alpha} |\lambda_\alpha(\beta)| = \|\lambda_\alpha\|_1$$

whence, as we obtained above for dQIs,

$$\|Q\|_\infty \le \max_{\alpha \in A} \|\lambda_\alpha\|_1 = \nu_1^*(Q).$$

As emphasized by de Boor (see e.g. [5], chapter XII), a QI defined on non uniform partitions has to be *uniformly bounded independently of the partition* (abbr. UB) in order to be interesting for applications. Therefore, the aim of this paper is to define some families of dQIs satisfying this property and having the *smallest possible norm*. As in general it is difficult to minimize the true norm of the operator, *we have chosen to solve the minimization problems* defined above. A further paper [25] will develop the case of iQIs on nonuniform partitions. A few results are given in [24].

The paper extends some results of [1][13] and is organized as follows. We first recall some "classical" QIs of various types and we verify that they are UB. Then we define and study several families of discrete and integral

QIs, depending on a finite number of parameters, for which we can find $\nu_1^*(Q)$. We show that this problem has always a solution (in general non unique). We give more specific examples for quadratic and cubic splines. Of particular interest are the results of theorems 3,5 and 6 where we show that some families of dQIs are uniformly bounded independently of the partition. Finally, we briefly give some applications to the approximation of functions, to quadrature formulas and to pseudo-spectral methods (see e.g. [12], [29]). A parallel study of spline QIs is done in [2] for uniform partitions of the real line and in [3] for some uniform triangulations of the plane.

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§2. Notations

We shall use classical B-splines of degree m on a bounded interval $I = [a, b]$ or on $I = \mathbb{R}$. For the sake of simplicity, in the case $I = \mathbb{R}$, we take an increasing sequence of knots $T = \{t_i, i \in \mathbb{Z}\}$. In the case $I = [a, b]$, we take the usual sequence T of knots defined by (see [5][11][19][28]):

$$a = t_{-m} = \ldots = t_0, \quad b = t_n = \ldots = t_{n+m}$$

$$a < t_1 < t_2 < \ldots < t_{n-1} < b$$

For $J = \{0, \ldots, n + m - 1\}$, the family of B-splines $\{B_j, j \in J\}$, with support $\Sigma_j = [t_{j-m}, t_{j+1}]$ is a basis of the space $S_m(I, T)$ of splines of degree m on the interval I endowed with the partition T. These B-splines form a partition of unity, i.e. $\sum_{j\in J} B_j = 1$. We denote $h_i = t_i - t_{i-1}$ for all indices i.
Let $\mathbb{N}_m = \mathbb{N} \cap [0, m - 1]$ and $T_j = \{t_{j-r}, r \in \mathbb{N}_m\}$: we recall that the *elementary symmetric functions* $\sigma_l(T)$ of the m variables in T_j are defined by $\sigma_0(T_j) = 0$ and for $1 \le l \le m$, by

$$\sigma_l(T_j) = \sum_{0\le r_1<r_2<\ldots<r_l\le m-1} t_{j-r_1} t_{j-r_2} \ldots t_{j-r_l}.$$

Let $C_m^l = \frac{m!}{l!(m-l)!}$ be the binomial coefficients, then the monomials $e_l(x) = x^l$ can be written $e_l = \sum_{i\in J} \theta_i^{(l)} B_i$, with $\theta_i^{(l)} = \sigma_l(T_i)/C_m^l$, for $0 \le l \le m$. This is a direct consequence of Marsden's identity ([3], chapter IX).
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§3. Differential QIs

For all $j \in J$, we define $\psi_j(t) = \prod_{r\in\mathbb{N}_m}(t_{j-r} - t)$ (thus $\psi_j \in \mathbb{P}_m$ for all $j \in J$). From de Boor and Fix [7] or de Boor ([5], chapter IX), we know that for any $\tau \in \Sigma_j$, the functionals

$$\lambda_j(f) = \frac{1}{m!} \sum_{l=0}^{m} (-1)^{m-l} D^{m-l} \psi_j(\tau) D^l f(\tau)$$

are dual functionals of B-splines, i.e. they satisfy, for all pairs $(i,j) \in J \times J$

$$\lambda_j(B_i) = \delta_{ij}.$$

Therefore the *differential quasi-interpolant* (abbr. DQI)

$$Qf = \sum_{j \in J} \lambda_j(f) B_j$$

satisfies $QB_j = B_j$ for all $j \in J$, i.e. Q is a *projector* on the space $S_m(I,T)$. In practice, it is interesting to choose $\tau = \theta_j = \frac{1}{m} \sum_{s \in \mathbb{N}_m} t_{j-s} = s_1(T_j) = \theta_j^{(1)}$. However, the computation of $\lambda_j(f)$ needs the evaluation all derivatives of polynomials ψ_j. Another method consists in writing Q in the form

$$Qf = \sum_{i \in J} \tilde{\lambda}_i(f) B_i,$$

whose coefficient functionals are defined by

$$\tilde{\lambda}_i(f) = \sum_{l=0}^{m} a_l(\theta_i) \frac{D^l f(\theta_i)}{l!},$$

and to impose that Q be exact on monomials of degree at most m

$$Qe_k = e_k \ \ for \ \ 0 \le k \le m.$$

Setting $\alpha_l(\theta_i) = \theta_i^{-l} a_l(\theta_i)$ and $\beta_s(\theta_i) = \theta_i^{-s} \theta_i^{(s)}$, we obtain the following system of linear equations, for $0 \le s \le m$:

$$\sum_{l=0}^{s} C_s^l \alpha_l(\theta_i) = \beta_s(\theta_i).$$

The solutions of this system are given by

$$\alpha_s(\theta_i) = \sum_{l=0}^{s} (-1)^{s-l} C_s^l \beta_l(\theta_i).$$

Thus we finally obtain

Theorem 1. *The coefficients of the differential forms* $\{\tilde{\lambda}_i(f), i \in J\}$, *are given, for* $0 \le s \le m$, *by*

$$a_s(\theta_i) = \sum_{l=0}^{s} (-1)^{s-l} C_s^l \theta_i^{s-l} \theta_i^{(l)}.$$

However, these DQIs need the values of derivatives of f, so they are not very easy to use in applications and we will not study them any more. Let us only give examples of quadratic and cubic DQIs.

Example 1: *Quadratic spline DQIs* (see also section 4 below). In de Boor's form, we have for $\tau = \theta_j$: $\lambda_j(f) = f(\theta_j) - \frac{1}{2}(\theta_j^2 - \theta_j^{(2)})D^2 f(\theta_j)$, where $\theta_j = \frac{1}{2}(t_{j-1} + t_j)$ and $\theta_j^{(2)} = t_{j-1}t_j$, whence $\theta_j^2 - \theta_j^{(2)} = \frac{1}{4}(t_{j-1} - t_j)^2$ and finally $\lambda_j(f) = f(\theta_j) - \frac{1}{8}h_i^2 D^2 f(\theta_j)$. Theorem 1 gives $a_0(\theta_j) = 1, a_1(\theta_j) = 0, a_2(\theta_j) = \theta_j^2 - \theta_j^{(2)}$, hence $\tilde{\lambda}_j(f) = \lambda_j(f)$.

Example 2: *Cubic spline DQIs*. In de Boor's form, we have for $\tau = \theta_j$: $\lambda_j(f) = f(\theta_j) - \frac{1}{2}(\theta_j^2 - \theta_j^{(2)})D^2 f(\theta_j) - \frac{1}{6}\psi_j(\theta_j)D^3 f(\theta_j)$, with $\theta_j = \frac{1}{3}(t_{j-2} + t_{j-1} + t_j)$ and $\theta_j^{(2)} = \frac{1}{3}(t_{j-2}t_{j-1} + t_{j-1}t_j + t_{j-2}t_j)$, whence $\theta_j^2 - \theta_j^{(2)} = \frac{1}{9}(h_{i-1}^2 + h_{i-1}h_i + h_i^2)$ and $\psi_j(\theta_j) = (t_j - \theta_j)(t_{j-1} - \theta_j)(t_{j-2} - \theta_j)$. Finally, we obtain $\lambda_j(f) = f(\theta_j) - \frac{1}{18}(h_{i-1}^2 + h_{i-1}h_i + h_i^2)D^2 f(\theta_j) - \frac{1}{162}(2h_{i-1} + h_i)(h_i - h_{i-1})(h_{i-1} + 2h_i)D^3 f(\theta_j)$. Theorem 1 gives $a_0(\theta_j) = 1, a_1(\theta_j) = 0, a_2(\theta_j) = \theta_j^2 - \theta_j^{(2)}, a_3(\theta_j) = \theta_j^{(3)} - 3\theta_j\theta^{(2)} + 2\theta_j^3 = \frac{1}{27}(2h_{i-1} + h_i)(h_i - h_{i-1})(h_{i-1} + 2h_i)$, whence $\tilde{\lambda}_j(f) = \lambda_j(f)$.

§4. Uniformly bounded discrete QIs exact on $\mathbb{P}_2$

It is now possible to derive *discrete* QIs from the preceding DQIs by replacing the values of derivatives $D^l f(\theta_i)/l!$ of f by divided differences at the points θ_r lying in Σ_i. Doing this, we loose the property of projection on $S_m(I, T)$. However, by choosing conveniently the divided differences, we can obtain some families of dQIs which are UB and exact on specific subspaces of polynomials.

Let us construct for example a family of dQIs of degree m which are *exact on* $\mathbb{P}_2$. We start from functionals which are truncations at order 2 of those of the preceding section:

$$\lambda_j^{(2)}(f) = \frac{1}{m!}\sum_{l=0}^{2}(-1)^{m-l}D^{m-l}\psi_j(\tau)D^l f(\tau).$$

As $\psi_j(t)$ is of degree m, we obtain successively $D^m\psi_j(\tau) = (-1)^m m!$, $D^{m-1}\psi_j(\tau) = (-1)^m m!(\tau - \theta_j)$, $D^{m-2}\psi_j(\tau) = \frac{1}{2}(-1)^m m!(\tau^2 - 2\theta_j t + \theta_j^{(2)})$. More specifically, taking $\tau = \theta_j$, we get

$$D^{m-1}\psi_j(\theta_j) = 0, \quad D^{m-2}\psi_j(\theta_j) = \frac{1}{2}(-1)^m m!(\theta_j^{(2)} - \theta_j^2)$$

and we obtain the DQI exact on $\mathbb{P}_2$

$$Q_2 f = \sum_{j\in J}\lambda_j^{(2)}(f)B_j,$$

whose coefficient functionals are given by

$$\lambda_j^{(2)}(f) = f(\theta_j) - \frac{1}{2}(\theta_j^2 - \theta_j^{(2)})D^2 f(\theta_j).$$

We recall the expansion (se e.g.[9][17]):

$$\bar{\theta}_j^{(2)} = \theta_j^2 - \theta_j^{(2)} = \frac{1}{m^2(m-1)} \sum_{(r,s)\in\mathbb{N}_m^2, r<s} (t_{j-r} - t_{j-s})^2 > 0.$$

On the other hand, $\frac{1}{2}D^2 f(\theta_j)$ coincide on the space $\mathbb{P}_2$ with the second order divided difference $[\theta_{j-1}, \theta_j, \theta_{j+1}]f$, therefore the dQI defined by

$$Q_2^* f = \sum_{j\in J} \mu_j^{(2)}(f) B_j,$$

with coefficient functionals

$$\mu_j^{(2)}(f) = f(\theta_j) - \bar{\theta}_j^{(2)}[\theta_{j-1}, \theta_j, \theta_{j+1}]f,$$

is also exact on $\mathbb{P}_2$. Moreover, one can write

$$\mu_i^{(2)}(f) = a_i f_{i-1} + b_i f_i + c_i f_{i+1}$$

with $a_i = -\bar{\theta}_i^{(2)}/\Delta\theta_{i-1}(\Delta\theta_{i-1} + \Delta\theta_i)$, $b_i = 1 + \bar{\theta}_i^{(2)}/\Delta\theta_{i-1}\Delta\theta_i$, $c_i = -\bar{\theta}_i^{(2)}/\Delta\theta_i(\Delta\theta_{i-1} + \Delta\theta_i)$. So, according to the introduction

$$\|Q_2^*\|_\infty \le \max_{i\in J}(|a_i| + |b_i| + |c_i|) \le 1 + 2\max_{i\in J} \bar{\theta}_i^{(2)}/\Delta\theta_{i-1}\Delta\theta_i.$$

The following theorem extends a result given for quadratic splines in [13][22][23].

Theorem 2. *For any degree m, the dQIs Q_2^* are uniformly bounded. More specifically, for all partitions of I:*

$$\|Q_2^*\|_\infty \le [\frac{1}{2}(m+4)]$$

Proof: We only give the proof for $m = 2k+1$, the case $m = 2k$ being similar. For the sake of simplicity, we take $j = k$, i.e. we shall determine an upper bound of the ratio

$$N_k/D_k = \bar{\theta}_k^{(2)}/\Delta\theta_{k-1}\Delta\theta_k$$

with

$$N_k = \bar{\theta}_k^{(2)} = \frac{1}{m^2(m-1)} \sum_{1\le r<s\le m} (t_r - t_s)^2.$$

Setting $H = \sum_{i=1}^{m-1} h_i$, then we get a lower bound for the denominator

$$D_k = \frac{1}{m^2}(t_{m-1} - t_0)(t_m - t_1) = \frac{1}{m^2}(h_0 + H)(H + h_{m-1}) \ge \frac{H^2}{m^2}$$

The numerator N_k is composed of $k-1$ pairs of sums (S_p, S'_p)

$$S_p = \sum_{s-r=p} (t_r - t_s)^2, \;\; S'_p = \sum_{s-r=k+p-1} (t_r - t_s)^2,$$

for $1 \le p \le k-1$. Both sums contain at most p times the terms h_i^2 and $2h_ih_j$ $(i \ne j)$, hence we can write $S_p + S'_p \le 2pH^2$, which implies

$$N_k \le \frac{2H^2}{(m-1)^2(m-2)}(1 + 2 + \ldots k - 1) = \frac{kS^2}{2(m-1)^2},$$

so, we get

$$N_k/D_k \le k/2,$$

and finally, for $m = 2k+1$ odd

$$\|Q_2^*\|_\infty \le k + 2 = \frac{1}{2}(m+3) = [\frac{1}{2}(m+4)]$$

For $m = 2k$, we obtain respectively $D_k \ge \frac{S^2}{4k^2}$ and $N_k \le \frac{S^2}{4(2k-1)}$, whence $N_k/D_k \le \frac{k^2}{2k-1}$, and finally for $m = 2k$ even

$$\|Q_2^*\| \le k + 2 = \frac{1}{2}(m+4) = [\frac{1}{2}(m+4)]$$

□

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§6. Existence and characterization of near-best discrete QIs

6.1. Existence of near-best dQIs

We consider the following family of dQIs defined, for the sake of simplicity, on $I = \mathbb{R}$ endowed with an arbitrary non-uniform increasing sequence of knots $T = \{t_i; i \in \mathbb{Z}\}$,

$$Qf = Q_{p,q}f = \sum_{i \in \mathbb{Z}} \mu_i(f) B_i.$$

Their coefficient functionals depend on $2p+1$ parameters, with $p \geq m$

$$\mu_i(f) = \sum_{s=-p}^{p} \lambda_i(s) f(\theta_{i+s}),$$

and they are exact on the space $\mathbb{P}_q$, where $q \leq \min(m, 2p)$. The latter condition is equivalent to $Qe_r = e_r$ for all monomials of degrees $0 \leq r \leq q$. It implies that for all indices i, the parameters $\lambda_i(s)$ satisfy the system of $q+1$ linear equations:

$$\sum_{s=-p}^{p} \lambda_i(s) \theta_{i+s}^r = \theta_i^{(r)}, \quad 0 \leq r \leq q.$$

The matrix $V_i \in \mathbb{R}^{(q+1)\times(2p+1)}$ of this system, with coefficients $V_i(r,s) = \theta_{i+s}^r$, is a Vandermonde matrix of maximal rank $q+1$, therefore there are $2p-q$ *free parameters*. Denoting $b_i \in \mathbb{R}^{q+1}$ the vector in the right hand side, with components $b_i(r) = \theta_i^{(r)}, \quad 0 \leq r \leq q$, we consider the sequence of minimization problems, for $i \in \mathbb{Z}$:

$$\min \|\lambda_i\|_1, \quad V_i \lambda_i = b_i.$$

We have seen in the introduction that $\nu_1^*(Q) = \max_{i \in \mathbb{Z}} \min \|\lambda_i\|_1$ is an upper bound of $\|Q_q\|_\infty$ which is easier to evaluate than the true norm of the dQI.

Theorem 3. *The above minimization problems have always solutions, which, in general, are non unique.*

Proof: The *objective function being convex* and *the domains being affine subspaces*, these classical optimization problems have always solutions, in general non unique.

We postpone to sections 7 and 8 the computation of some optimal solutions in the case $q = 2$.

6.2. Characterization of optimal solutions

For $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$, let us consider the l_1-minimization problem

$$(1) \quad \min \|r(a)\|_1, \quad r(a) = b - Aa.$$

We recall the characterization of optimal solutions for l_1-problems given in [30], chapter 6. Define the sets

$$Z(a) = \{1 \leq i \leq m |\ r_i(a) = 0\}$$

$$V(a) = \{v \in \mathbb{R}^m ; \|v\|_\infty \leq 1,\ v_i = sgn(r_i(a))\ for\ i \notin Z(a)\}$$

Theorem 4. *a^* is a solution of (1) if and only if there exists a vector $v^* \in V(a^*)$ satisfying $A^T v^* = 0$.*

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§7. A general family of spline discrete QIs exact on $\mathbb{P}_2$

In this section, we restrict our study to the subfamily of spline dQIs which are *exact on* $\mathbb{P}_2$, i.e; we choose $q = 2$. We shall try to characterize optimal solutions in the sense of theorem 3 with the help of theorem 4. Let

$$Q_{p,2} f = \sum_{i \in \mathbb{Z}} \mu_i(f) B_i,$$

where the coefficient functionals depend on $2p + 1$ parameters

$$\mu_i(f) = \sum_{r=-p}^{p} \lambda_i(r) f(\theta_{i+r}).$$

We shall need the following sets of indices

$$\bar{K} = \{-p, \ldots, p\}, \quad K^* = \{-p, 0, p\}, \quad K = \bar{K} \backslash K^*.$$

$$K = K_1 \cup K_2, \quad K_1 = \{-p+1, \ldots, -1\}, \quad K_2 = \{1, \ldots, p-1\}.$$

The three equations expressing the exactness of $Q_{p,2}$ on $\mathbb{P}_2$ can be written

$$\lambda_i(-p) + \lambda_i(0) + \lambda_i(p) = 1 - \sum_{r \in K} \lambda_i(r)$$

$$\theta_{i-p} \lambda_i(-p) + \theta_i \lambda_i(0) + \theta_{i+p} \lambda_i(p) = \theta_i - \sum_{r \in K} \theta_r \lambda_i(r)$$

$$\theta_{i-p}^2 \lambda_i(-p) + \theta_i^2 \lambda_i(0) + \theta_{i+p}^2 \lambda_i(p) = \theta_i^{(2)} - \sum_{r \in K} \theta_r^2 \lambda_i(r)$$

This system has a positive Vandermonde determinant

$$V_i = V(\theta_{i-p}, \theta_i, \theta_{i+p}) = (\theta_i - \theta_{i-p})(\theta_{i+p} - \theta_i)(\theta_{i+p} - \theta_{i-p}).$$

Let us denote by

$$(\lambda_i^*(-p), \lambda_i^*(0), \lambda_i^*(p))$$

the unique solution of the above system with the right-hand side obtained by taking $\lambda_i(r) = 0$ for all $r \in K$. Using Cramer's rule and the determinants $W_i(s)$ obtained by replacing the column of θ_{i+s} in V_i by this rhs, we obtain

$$\lambda_i^*(-p) = W_i(-p)/V_i, \quad \lambda_i^*(0) = W_i(0)/V_i, \quad \lambda_i^*(p) = W_i(p)/V_i.$$

Then we can express the general solution of the above system in the form

$$\lambda_i(-p) = \lambda_i^*(-p) - \sum_{r \in K_1} \alpha_r \lambda_i(r) + \sum_{s \in K_2} \alpha_s \lambda_i(s)$$

$$\lambda_i(0) = \lambda_i^*(0) - \sum_{r \in K_1} \beta_r \lambda_i(r) - \sum_{s \in K_2} \beta_s \lambda_i(s)$$

$$\lambda_i(p) = \lambda_i^*(p) + \sum_{r \in K_1} \gamma_r \lambda_i(r) - \sum_{s \in K_2} \gamma_s \lambda_i(s)$$

The various coefficients are quotients of Vandermonde determinants

$$\alpha_r = V(\theta_r, \theta_i, \theta_{i+p})/V_i, \quad \alpha_s = V(\theta_i, \theta_s, \theta_{i+p})/V_i,$$
$$\beta_r = V(\theta_{i-p}, \theta_r, \theta_{i+p})/V_i, \quad \beta_s = V(\theta_{i-p}, \theta_s, \theta_{i+p})/V_i,$$
$$\gamma_r = V(\theta_{i-p}, \theta_r, \theta_i)/V_i, \quad \gamma_s = V(\theta_{i-p}, \theta_i, \theta_s)/V_i.$$

We denote by $Q_{p,2}^*$ the spline dQI whose coefficient functionals are

$$\mu_i^*(f) = \lambda_i^*(-p) f(\theta_{i-p}) + \lambda_i^*(0) f(\theta_i) + \lambda_i^*(p) f(\theta_{i+p}).$$

In that case, an upper bound of the norm is $\max_{i \in \mathbb{Z}} \nu_i^*$ where

$$\nu_i^* = |\lambda_i^*(-p)| + |\lambda_i^*(0)| + |\lambda_i^*(p)|$$

Theorem 5. *For all $p \geq m =$ degree of the spline, the infinite norms of the spline dQIs $Q_{p,2}^*$ are uniformly bounded by $\frac{m+1}{m-1}$. This bound is independent of p and of the sequence of knots T.*

Proof: We have to find a good upper bound of

$$\nu_i^* = |\lambda_i^*(-p)| + |\lambda_i^*(0)| + |\lambda_i^*(p)|$$

where, expanding the determinants, we have

$$\lambda_i^*(-p) = -\bar{\theta}_i^{(2)}/(\theta_{i+p} - \theta_{i-p})((\theta_i - \theta_{i-p})$$

$$\lambda_i^*(0) = 1 + \bar{\theta}_i^{(2)}/(\theta_{i+p} - \theta_i)((\theta_i - \theta_{i-p})$$

$$\lambda_i^*(p) = -\bar{\theta}_i^{(2)}/(\theta_{i+p} - \theta_{i-p})((\theta_{i+p} - \theta_i).$$

We recall that $\theta_i = \frac{1}{m}\sum_{r=0}^{m-1} t_{i-r}$ and

$$\bar{\theta}_i^{(2)} = \frac{1}{m^2(m-1)} \sum_{(r,s)\in\mathbb{N}_m, r<s} (t_{i-r} - t_{i-s})^2 = \frac{S_1}{m^2(m-1)}$$

We first compute

$$\theta_i - \theta_{i-p} = \frac{1}{m}\sum_{r\in\mathbb{N}_m}(t_{i-r} - t_{i-p-r}) = \frac{1}{m}\sum_{r\in\mathbb{N}_m}\sum_{k=0}^{p} h_{i-k-r+1} = S_2/m.$$

$$\theta_{i+p} - \theta_i = \frac{1}{m}\sum_{r\in\mathbb{N}_m}(t_{i+p-r} - t_{i-r}) = \frac{1}{m}\sum_{r\in\mathbb{N}_m}\sum_{k=0}^{p} h_{i+k-r+1} = S_3/m,$$

The proof being essentially the same for all $p \geq m$ and for all $i \in \mathbb{Z}$, we can restrict our study to the cases $p = m$ and $i = m-1$. In that case, we get

$$S_2 = mh_{i-m+1} + \sum_{k=1}^{m-1} k(h+h) \geq S_2' = h_1 + 2h_2 + \ldots + (m-1)h_{m-1},$$

$$S_3 = mh_{i-m+1} + \sum_{k=1}^{m-1} k(h+h) \geq S_3' = (m-1)h_1 + (m-2)h_2 + \ldots + 2h_{m-2} + h_{m-1}.$$

Denoting, for $1 \leq k \leq m-1$

$$s_k = h_1 + \ldots + h_k,$$

and $s = s_{m-1}$, we get

$$S_2' = s_1 + s_2 + \ldots + s_{m-1}, \quad S_3' = s + (s - s_1) + (s - s_2) + \ldots (s - s_{m-2}) = ms - S_2'$$

whence

$$S_2 S_3 \geq S_2' S_3' = ms(s_1 + s_2 + \ldots + s_{m-1}) - (s_1 + s_2 + \ldots + s_{m-1})^2.$$

Now, we come back to S_1 and we shall prove that $S_1 \leq S_2'S_3' \leq S_2S_3$. S_1 can be written under the form

$$S_1 = \sum_{j=1}^{m-1} \sum h_{i-r+j} = \sum_{i=1}^{m-1} s_i^2 + \sum_{j=1}^{m-1} \sum_{i=j+1}^{m} (s_i - s_j)^2,$$

from which we deduce

$$S_1 = (m-1) \sum_{i=1}^{m-1} s_i^2 - 2 \sum_{j=1}^{m-1} s_j \sum_{i=j+1}^{m} s_i.$$

Moreover, for all $1 \leq i \leq m-1$, we have

$$(m-1)s_i^2 = ms_i^2 - s_i^2 \leq ms_i s_{m-1} - s_i^2$$

therefore, we obtain the result

$$S_1 \leq (m-1) \sum_{i=1}^{m-1} s_i^2 \leq ms_{m-1} \sum_{i=1}^{m-1} s_i - \sum_{i=1}^{m-1} s_i^2 = S_2'S_3' \leq S_2S_3.$$

Finally, for all $i \in \mathbb{Z}$, we have

$$\nu_i^* = 1 + \frac{2}{m-1} \frac{S_1}{S_2S_3} \leq 1 + \frac{2}{m-1} = \frac{m+1}{m-1},$$

whence $\|Q_{p,2}^*\|_\infty \leq \max_{i\in\mathbb{Z}} \nu_i^* \leq \frac{m+1}{m-1}$. $\square$

In the next section, we prove that the QIs $Q_{p,2}^*$ are near-best in the sense of section 6.

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§8. A family of near-best spline discrete QIs

For dQIs $Q_{p,2}$ depending on $p \geq m$ parameters, the coefficients (see proof of theorem 5) are given by

$$\lambda_i^*(-p) = -\bar{\theta}_i^{(2)}/(\theta_{i+p} - \theta_{i-p})((\theta_i - \theta_{i-p})$$

$$\lambda_i^*(0) = 1 + \bar{\theta}_i^{(2)}/(\theta_{i+p} - \theta_i)((\theta_i - \theta_{i-p})$$

$$\lambda_i^*(p) = -\bar{\theta}_i^{(2)}/(\theta_{i+p} - \theta_{i-p})((\theta_{i+p} - \theta_i).$$

Now, let us write the minimization problem of section 7 in Watson's form. Denote

$$\tilde{\lambda}_i = (\lambda_i(-p+1), \ldots, \lambda_i(-1), \lambda_i(1), \ldots, lambda_i(p-1))^T \in \mathbb{R}^{2p-2}$$

$$\lambda_i^* = (\lambda_i^*(-p), 0, \dots, \lambda_i^*(0), 0, \dots, \lambda_i^*(p))^T \in \mathbb{R}^{2p+1}$$

Let $A_i \in \mathbb{R}^{(2p+1)\times(2p-1)}$ be the matrix with the following coefficients (notations of section 7)

$$For\ r \in K_1 : A_i(-p, r) = \alpha_r,\ A_i(0, r) = \beta_r,\ A_i(p, r) = -\gamma_r,$$

$$For\ s \in K_2 : A_i(-p, s) = -\alpha_s,\ A_i(0, s) = \beta_s,\ A_i(p, s) = \gamma_s,$$

$$For\ r \in K_1 :\ A_i(r, r') = 0,\ r' \neq r,\ A_i(r, r) = -1, \quad A_i(r, s) = 0,\ s \in K_2,$$

$$For\ s \in K_2 :\ A_i(s, r) = 0,\ r \in K_1, \quad A_i(s, s) = -1,\ A_i(s, s') = 0,\ s' \neq s.$$

Then, using these notations, we can write

$$\|\lambda_i\|_1 = \|\lambda_i^* - A_i\tilde{\lambda}_i\|_1$$

Theorem 6. *Assume that the sequence of knots T satisfies, for all $i \in \mathbb{Z}$, the following properties*

$$\theta_{i-1} + \theta_i \leq \theta_{i-p} + \theta_{i+p} \leq \theta_i + \theta_{i+1},$$

then, for all $i \in \mathbb{Z}$, λ_i^* is an optimal solution of the local minimization problem $\min \|\lambda_i\|_1$. *Thus, for all $p \geq m$, the spline dQIs $Q_{p,2}^*$ are near-best and their infinite norms are uniformly bounded by $\frac{m+1}{m-1}$. This bound is independent of p and of the sequence of knots T.*

Proof: According to Watson's theorem, we must find a vector $v^* \in \mathbb{R}^{2p+1}$ satisfying

$$\|v^*\|_\infty \leq 1, \quad A_i^T v^* = 0, \quad v^*(r) = sgn(\lambda_i^*(r))\ \ for\ \ r = -p, 0, p.$$

Let us choose

$$v^*(-p) = -1, v^*(0) = 1, v^*(p) = -1,$$

$$v^*(r) = -\alpha_r + \beta_r + \gamma_r, \ \ for\ \ r \in K_1,$$

$$v^*(s) = -\alpha_s + \beta_s + \gamma_s, \ \ for\ \ s \in K_2.$$

Then it is easy to verify that the equations $A_i^T v^* = 0$ are satisfied. Moreover, the above expressions of $\lambda_i^*(r)$ for $r = -p, 0, p$ with $\bar{\theta}_i^{(2)} > 0$ imply that $sgn(v^*(r)) = sgn(\lambda_i^*(r))$ for $r = -p, 0, p$. It only remains to prove that, for $(r, s) \in K_1 \times K_2$

$$|v^*(r)| = |-\alpha_r + \beta_r + \gamma_r| \leq 1, \quad |v^*(s)| = |\alpha_s + \beta_s - \gamma_s| \leq 1.$$

As $\beta_r = 1 - \alpha_r + \gamma_r$ for $r \in K_1$ and $\beta_s = 1 + \alpha_s - \gamma_s$ for $s \in K_2$, it is equivalent to prove

$$0 \le \alpha_r - \gamma_r \le 1, \quad 0 \le \gamma_s - \alpha_s \le 1, \;\; for \;\; (r,s) \in K_1 \times K_2$$

We only detail the proof for $r \in K_1$, that for $s \in K_2$ being quite similar. Using the Vandermonde determinants, we get

$$\alpha_r - \gamma_r = V_i^{-1}(\theta_i - \theta_r)(\theta_{i+p} - \theta_{i-p})[(\theta_{i+p} + \theta_{i-p}) - (\theta_r + \theta_i)],$$

As $\theta_i - \theta_r \ge 0$ and $\theta_{i+p} - \theta_{i-p} \ge 0$, we shall have $\alpha_r - \gamma_r \ge 0$ if and only if

$$\theta_r + \theta_i \le \theta_{i+p} + \theta_{i-p}$$

for all $r \in K_1$. However, since we have $\theta_r + \theta_i \le \theta_{i-1} + \theta_i$, there only remains the unique condition

$$\theta_{i-1} + \theta_i \le \theta_{i-p} + \theta_{i+p}.$$

The other inequality $\alpha_r - \gamma_r \le 1$ can be written

$$(\theta_i - \theta_r)[(\theta_{i+p} + \theta_{i-p}) - (\theta_r + \theta_i)] \le (\theta_i - \theta_{i-p})(\theta_{i+p} - \theta_i)$$

Setting $\delta_1 = \theta_r - \theta_{i-p}$, $\delta_2 = \theta_i - \theta_r$, and $\delta_3 = \theta_{i+p} - \theta_i$, the latter inequality can be written

$$\delta_2(\delta_3 - \delta_1) \le \delta_3(\delta_2 + \delta_1), \;\; or \;\; \delta_1(\delta_2 + \delta_3) \ge 0$$

which is obviously satisfied. For $s \in K_2$, the inequalities $0 \le \gamma_s - \alpha_s \le 1$ are satisfied if and only if

$$\theta_{i-p} + \theta_{i+p} \le \theta_i + \theta_{i+1},$$

whence the conditions on the sequence of knots. $\square$

Remark. Theorem 6 imposes some conditions on the sequence of knots. For quadratic splines, we have studied arithmetic and geometric sequences: in both cases, the higher is p, the stronger are the conditions and for $p \to +\infty$, T is closer and closer to a uniform sequence.
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§9. Some applications

9.1. Approximation of functions

When a spline dQI Q is uniformly bounded independently of the partition, we can apply a classical result in approximation theory (see [5], Th 22, and [11], chapters 2 and 5):

$$\|Qf - f\|_\infty \leq (1 + \|Q\|_\infty) d_\infty(f, \mathcal{S})$$

where $\mathcal{S}$ is the space of splines. In particular, when Q is exact on the space $\mathbb{P}_m$, then for $f \in C^{m+1}(I)$, one has

$$\|Qf - f\|_\infty \leq Ch^{m+1}\|f^{m+1}\|_\infty$$

for some constant C which does not depend on the given partition. Therefore spline dQIs give the best possible approximation order. More detailed results on error bounds are given in [1],[13] and [23].

9.2. Quadrature formulas

Approximating $\int_I f$ by $\int_I Q_2^*(f)$, where $Q_2^*(f)$ is the quadratic spline dQI of section 8, gives rise to an interesting quadrature formula

$$\int_I Q_2^*(f) = f_0 \int_I B_0 + \sum_{i=1}^{n} \mu_i(f) \int_I B_i + f_{n+1} \int_I B_{n+1}$$

As it is well known, $\int_I B_0 = \frac{h_1}{3}$, $\int_I B_{n+1} = \frac{h_n}{3}$ and $\int_I B_i = \frac{h_{i-1}+h_i+h_{i+1}}{3}$ for $1 \leq i \leq n$. This formula is exact on $\mathbb{P}_2$, but in the case of a uniform partition,(see [21]), it is exact on $\mathbb{P}_3$ and provides an interesting complementary formula to Simpson's rule in the sense that, in general, errors for both formulas have opposite signs. This will be detailed in another paper, together with applications to integral equations.

9.3. Pseudo-spectral methods

One can approximate the first derivatives of a given function f at the data sites

$$\Theta_n = \{\theta_0 = t_0, \;\; \theta_i = \frac{1}{2}(t_{i-1} + t_i), \;\; for \;\; 1 \leq i \leq n, \;\; \theta_{n+1} = t_n\}.$$

by the derivatives of the quadratic spline dQI of section 8.1

$$Q_2^* f = f(t_0)B_0 + \sum_{i=1}^{n} \mu_i(f) B_i + f(t_n) B_{n+1}.$$

For interior points θ_i, $\;3 \leq i \leq n-2$, we obtain the general formula

$$(Q_2^* f)'(\theta_i) = \mu_{i-1}(f) B'_{i-1}(\theta_i) + \mu_i(f) B'_i(\theta_i) + \mu_{i+1}(f) B'_{i+1}(\theta_i)$$

which can also be written, by setting $f_j = f(\theta_j)$:

$$(Q_2^* f)'(\theta_i) = \frac{1}{h_i}\{-\sigma_i a_{i-1} f_{i-2} + [-\sigma_i b_{i-1} + (\sigma_i - \sigma'_{i+1})a_i] f_{i-1}$$

$$[-\sigma_i c_{i-1} + (\sigma_i - \sigma'_{i+1})b_i + a_{i+1}] f_i + [(\sigma_i - \sigma'_{i+1})c_i + b_{i+1}] f_{i+1} + \sigma'_{i+1} c_{i+1} f_{i+2}\}$$

For the first indices $0 \le i \le 2$, the coefficients are modified according to the convention $h_0 = 0$, which gives $\sigma_0 = 0, \sigma'_0 = 1, \sigma_1 = 1$ and $\sigma'_1 = 0$. We thus obtain

$$(Q_2^* f)'(\theta_0) = \frac{2}{h_1}\{(a_1 - 1)f_0 + b_1 f_1 + c_1 f_2\}$$

$$(Q_2^* f)'(\theta_1) = \frac{1}{h_1}\{(\sigma_2 a_1 - 1)f_0 + [\sigma_2 b_1 + \sigma'_2 a_2] f_1 +$$

$$[\sigma_2 c_1 + \sigma'_2 b_2] f_2 + \sigma'_2 c_2 f_3\}$$

$$(Q_2^* f)'(\theta_2) = \frac{1}{h_2}\{-\sigma_2 a_1 f_0 + [-\sigma_2 b_1 + (\sigma_2 - sigma'_3)a_2] f_1 + [-\sigma_2 c_1 + (\sigma_2 - \sigma'_3)b_2 + \sigma'_3 a_3] f_2$$

$$+[(\sigma_2 - \sigma'_3)c_2 + \sigma'_3 b_3] f_3 + \sigma'_3 c_3 f_4\}$$

In the same way, for the last indices $n-1 \le i \le n+1$, the coefficients are modified according to the convention $h_{n+1} = 0$ and we obtain similar formulas for $(Q_2^* f)'(\theta_{n-1}), (Q_2^* f)'(\theta_n)$ and $(Q_2^* f)'(\theta_{n+1})$. In the case of a unifom partition, the formulas are given in [21]. Concerning error estimates, it is rather easy to verify that $(Q_2^* f)'(\theta_i) - f'(\theta_i) = O(h^2)$ where $h = \max_{1 \le i \le n} h_i$. A more detailed study will be done elsewhere. These results can be used in pseudo-spectral methods, as described for example in [12] and [29].

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